Chapter 2

Composite Systems and Entanglement

In the previous section we discussed a single two level system and while its pretty cool how much quantum physics can be discussed when looking at such a simple system.... there's only so far you can go. In this chapter, we extend the quantum formalism to analyze the behavior of quantum systems composed of many degrees of freedom. We will see that when the postulates of quantum mechanics are applied to systems of many particles, they give rise to interesting and counter-intuitive phenomena such as quantum entanglement.

2.1 State Space for Many Particles

Suppose we have two particles, labeled A and B. We know the state of the system comprising both particles, which we call AB, must be described by a vector in a complex vector space. The natural question to ask is, in what space does a generic state for the two particles, $|\psi_{AB}\rangle$, live? If we call $\mathcal{H}_A$ and $\mathcal{H}_B$ the vector (Hilbert) spaces in which the quantum states of the individual particles live, then it is a postulate of quantum mechanics that a generic state vector describing the combined system lives in a space

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

The symbol $\otimes$ refers to a tensor product, a mathematical operation that combines two vector (Hilbert) spaces to produce another one.

The meaning of the tensor product is more easily understood in terms of explicit basis vectors, in the case of discrete vector spaces. For this purpose, let us assume that $\mathcal{H}_A$ is spanned by a set of basis vectors $\{|\mu_1\rangle, |\mu_2\rangle, |\mu_3\rangle, \ldots |\mu_{n_A}\rangle\}$ and that $\mathcal{H}_B$ is spanned by a set of other basis vectors $\{|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle, \ldots |\nu_{n_B}\rangle\}$. Then, the vector space $\mathcal{H}_{AB}$ is by construction spanned by basis vectors consisting of all the pairwise combinations of the basis vectors of A and B, and the basis states of the composite system are written as

$$|\mu_i\rangle \otimes |\nu_j\rangle \quad \forall\, i \in [1, n_A], j \in [1, n_B].$$

We can see that the total number of basis states for the composite system is $n_A \times n_B$. All quantum states in $\mathcal{H}_{AB}$ can be written as linear combinations of the composite basis states:

$$|\psi_{AB}\rangle = \sum_{ij} c_{ij} |\mu_i\rangle \otimes |\nu_j\rangle = \sum_{ij} c_{ij} |\lambda_{ij}\rangle$$

with c_{ij} being some complex coefficients, and where we have defined the basis vectors $|\lambda_{ij}\rangle \equiv |\mu_i\rangle \otimes |\nu_j\rangle$.

In order to work with these states, we need to know how to perform inner products between states belonging to the tensor product space $\mathcal{H}_{AB}$. The inner product between two basis states is defined as

$$\langle\lambda_{ij}|\lambda_{kl}\rangle = (\langle\mu_i| \otimes \langle\nu_j|)(|\mu_k\rangle \otimes |\nu_l\rangle) \equiv \langle\mu_i|\mu_k\rangle\langle\nu_j|\nu_l\rangle = \delta_{ik}\delta_{jl}.$$

This definition is relatively easy to understand: the inner product is obtained as the product of the elementary (A or B) inner products. Also, it shows that the basis states of the composite system are orthogonal by construction. As a consequence, the inner product between two generic states of the composite system

$$|\phi\rangle = \sum_{ij} b_{ij}|\lambda_{ij}\rangle, \quad |\psi\rangle = \sum_{ij} c_{ij}|\lambda_{ij}\rangle,$$

reads

$$\langle\phi|\psi\rangle = \sum_{ij}\sum_{kl} b^*_{ij}c_{kl}\langle\lambda_{ij}|\lambda_{kl}\rangle = \sum_{ij} b^*_{ij}c_{ij}.$$

We also see that the basis states of the composite system satisfy the closure relation:

$$\sum_{ij} |\lambda_{ij}\rangle\langle\lambda_{ij}| = I.$$

Formally speaking, the tensor product satisfies all the intuitive properties you might expect from a product. For example, given a scalar a and two arbitrary vectors $|v\rangle \in \mathcal{H}_A$ and $|w\rangle \in \mathcal{H}_B$, we have

$$a(|v\rangle \otimes |w\rangle) = (a|v\rangle) \otimes |w\rangle = |v\rangle \otimes (a|w\rangle)\,.$$

It is also distributive:

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle,$$

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle.$$

Finally, the construction of the product state space can be generalized from the case of two particles to the case of many particles, A, B, C, ..., since the composite vector (Hilbert) space will be simply given by the tensor product of the individual state spaces

$$\mathcal{H}_{ABC...} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes H_C \ldots,$$

and in general, the resulting space will have a large dimension when we have many particles, since it is the product of the size of the individual dimensions

$$n_{ABC...} = n_A \times n_B \times n_C \times \ldots\ .$$

2.1.1 Example: Two Qubits

Let us see an example of this formalism in the case of two qubits, i.e., for that case that $\mathcal{H}_A$ and $\mathcal{H}_B$ are both vector spaces of dimension 2. As basis states of the individual spins, we take the eigenkets of σ_Z, thus the resulting tensor product space is given by the 4 states

$$|1\rangle_{AB} = |0\rangle_A \otimes |0\rangle_B$$
$$|2\rangle_{AB} = |0\rangle_A \otimes |1\rangle_B$$
$$|3\rangle_{AB} = |1\rangle_A \otimes |0\rangle_B$$
$$|4\rangle_{AB} = |1\rangle_A \otimes |1\rangle_B,$$

and a generic state of two qubits is written as

$$|\psi\rangle_{AB} = \sum_{k=1}^{4} c_k |k\rangle_{AB},$$

where, as always, by definition

$$c_k = \langle k|\psi\rangle.$$

For example, take

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right) = \frac{1}{\sqrt{2}} \left(|2\rangle - |3\rangle\right).$$

We can easily check that this is a physically valid state, since it is correctly normalized:

$$\langle\psi|\psi\rangle = \frac{1}{2} \left(\langle 2|2\rangle + \langle 3|3\rangle\right) = 1.$$

A note on notation! Writing out the composite state of $|\psi\rangle$ and $|\phi\rangle$ as $|\psi\rangle \otimes |\phi\rangle$ can feel a bit cumbersome. So we often don't bother to explicitly write out the $\otimes$ and instead write $|\psi\rangle|\phi\rangle$ or just $|\psi\phi\rangle$. That is,

- $|\psi\rangle \otimes |\phi\rangle$
- $|\psi\rangle|\phi\rangle$
- $|\psi, \phi\rangle$
- $|\psi\phi\rangle$

all mean the same thing! And you need to become comfortable switching between these notations.

2.2 Operators

So far, we have introduced the state space for a system of many particles, but we haven't talked about the operators that act on this space, and how they are related to the measurement process. If we have two operators T_A and T_B acting on the individual spaces, the resulting operator that acts on the product space is also written as a tensor product:

$$T_{AB} = T_A \otimes T_B,$$

where the resulting operator T_{AB} now acts on vectors in the space $H_A \otimes \mathcal{H}_B$. The composite operator acts as follows:

$$T_{AB}|\lambda_{ij}\rangle = (T_A \otimes T_B)(|\mu_i\rangle \otimes |\nu_j\rangle) \equiv (T_A|\mu_i\rangle) \otimes (T_B|\nu_j\rangle),$$

thus, quite naturally, each of the two operators in the product acts on the kets that belong to the respective vector spaces. As a special case, notice that if we are given only an operator that acts on one of the two subsystems, this is to be understood as

$$T'_{AB} = T_A \otimes I_B$$

if only T_A is given, and where I_B is the identity operator for subsystem B. Similarly,

$$T''_{AB} = I_A \otimes T_B,$$

if only T_B is given. As a result, it is easy to see that these two operators, acting non-trivially only on one of the two subsystems, commute since:

$$\begin{aligned} T''_{AB}T'_{AB}|\lambda_{ij}\rangle &= (I_A \otimes T_B)(T_A \otimes I_B)(|\mu_i\rangle \otimes |\nu_j\rangle) = (T_A|\mu_i\rangle) \otimes (T_B|\nu_j\rangle), \\ T'_{AB}T''_{AB}|\lambda_{ij}\rangle &= (T_A \otimes I_B)(I_A \otimes T_B)(|\mu_i\rangle \otimes |\nu_j\rangle) = (T_A|\mu_i\rangle) \otimes (T_B|\nu_j\rangle), \end{aligned}$$

thus

$$[T_A \otimes I_B, I_A \otimes T_B] = 0.$$

2.2.1 Example: Spin $\frac{1}{2}$ Operators

Let us give again an example for two qubits A and B. For concreteness, let's now suppose that the qubit represents a spin 1/2 particle. We write the spin z operator on the two individual systems as[1] $S_A^{(z)} = \frac{1}{2}Z_A$ where Z_A is the standard Pauli operator on system A such that

$$\begin{aligned} S_A^{(z)}|m\rangle_A &= m|m\rangle_A, \\ S_B^{(z)}|m'\rangle_B &= m'|m'\rangle_B, \end{aligned}$$

for $m, m' = \pm\frac{1}{2}$. It is then natural to define the total spin as the sum of these two operators. In order to do so, however, we need to recall that these operators are acting on different spaces, thus before summing them up we need to "upgrade" them to be good operators for the composite vector space. The total $S_{AB}^{(z)}$ operator reads:

$$S_{AB}^{(z)} = S_A^{(z)} \otimes I_B + I_A \otimes S_B^{(z)}.$$

It is then straightforward to see how this operator acts on a general state. For example, if we take a basis vector for the composite system, we have

$$\begin{aligned} S_{AB}^{(z)}\left(|m\rangle_A \otimes |m'\rangle_B\right) &= \left(S_A^{(z)} \otimes I_B + I_A \otimes S_B^{(z)}\right)\left(|m\rangle_A \otimes |m'\rangle_B\right) \\ &= \left(S_A^{(z)}|m\rangle_A\right) \otimes |m'\rangle_B + |m\rangle_A \otimes \left(S_B^{(z)}|m'\rangle_B\right) \\ &= m\left(|m\rangle_A \otimes |m'\rangle_B\right) + m'\left(|m\rangle_A \otimes |m'\rangle_B\right) \\ &= (m+m')\left(|m\rangle_A \otimes |m'\rangle_B\right). \end{aligned} \tag{2.1}$$

thus the composite state is an eigenket of the total spin, with an eigenvalue $(m+m')$ that is the sum of the individual eigenvalues.

Remember that a qubit can represent all sorts of different systems and so this maths applies more broadly. For example, if the qubit represents two energy levels of an atom with a Hamiltonian $H_A = \omega Z_A$ then $H_A \otimes I_B + I_A \otimes H_B$ would allow us to compute the total energy of two atoms.

[1] Remember, we work in nice tidy units such that $\hbar = 1$

A note on notation! In the case of a composite operator $H_A \otimes H_B$ you *cannot* drop the $\otimes$ (this is because $H_A H_B$ looks like you are multiplying the matrices) but its common to be lazy and drop identity operations. That is, write Z_A instead of $Z_A \otimes I_B$ or $Z \otimes I$. So, for example,

- $Z_A \otimes I_B + I_A \otimes Z_B$
- $Z \otimes I + I \otimes Z$
- $Z_A + Z_B$

mean the same thing. Again, you'll need to get comfortable switching notations.

2.2.2 Explicit matrix and vector representation of the tensor product

Generally you should try and stick to braket notation - this is typically simpler than writing out explicit matrix descriptions of states of multi-qubit systems. But sometimes it is helpful to visualise the composite vectors/operators explicitly. The basic idea behind the tensor product is to multiply a copy of the second matrix by each element of the first matrix in turn and so we have

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha \\ a\beta \\ b\alpha \\ b\beta \end{pmatrix}.$$

Note that, for example, the matrix representation of $|10\rangle$ is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

exactly what would be naively expected. An equivalent approach can be used for operators, e.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

2.3 Measurements

For the single-component case, recall that the measurement process in quantum mechanics works as follows. Consider a measurement operator $\hat{A}$ with eigenkets $|A_i\rangle$ and corresponding eigenvalues a_i. Without loss of generality, an arbitrary state $|\psi\rangle$ can be expressed in this basis:

$$|\psi\rangle = \sum_i \beta_i |A_i\rangle, \quad \text{where} \quad \beta_i = \langle A_i|\psi\rangle \in \mathbb{C}.$$

Measuring $|\psi\rangle$ under the operator $\hat{A}$ collapses the state into eigenket $|A_i\rangle$ with probability $P_i = |\beta_i|^2$, producing measurement result a_i.

In the case of a composite system, there are two kinds of measurements we can perform.

2.3.1 Global Measurement

In the first case, we measure an operator $T = T_A \otimes T_B$, thus intrinsically defined to act on the joint vector space, and in this sense corresponding to a measurement of the entire system AB. Similarly to the standard situation, then we can diagonalize the operator:

$$T|T_i\rangle = t_i|T_i\rangle,$$

in such a way that (assuming the operator has a non-degenerate spectrum)

$$|\psi\rangle = \sum_i |T_i\rangle\langle T_i|\psi\rangle,$$

thus a measurement will yield the state $|T_i\rangle$ with probability $P_i = |\langle T_i|\psi\rangle|^2$.

2.3.2 Partial Measurement

In the second case, we can measure an operator that is defined only on one of the two subsystems, for example T_A. In this sense, we are performing a partial measurement of the system, since we measure only the properties of one subpart, ignoring the rest of the system. We can rewrite a generic state of two particles as

$$|\psi\rangle = \sum_{ij} c_{ij}|T_{Ai}\rangle \otimes |T_{Bj}\rangle = \sum_i |T_{Ai}\rangle \otimes \left(\sum_j c_{ij}|T_{Bj}\rangle\right) = \sum_i |T_{Ai}\rangle \otimes |\phi_i^B\rangle,$$

where we have defined

$$|\phi_i^B\rangle = \sum_j c_{ij}|T_{Bj}\rangle.$$

This expression then allows us to get a better intuition about what happens when we measure only the first subsystem (A). In that case, assuming that we measure the operator T_A with eigenvalues t_{Ai}, it is postulated that after the measurement the system collapses into

$$|\psi_i'\rangle \propto |T_{Ai}\rangle \otimes |\phi_i^B\rangle.$$

The probability for this to happen is postulated to be

$$P_i = \langle\psi_i|(|T_{Ai}\rangle\langle T_{Ai}| \otimes I_B\rangle)|\psi_i\rangle$$

which is a generalization of what we have seen for the single particle case. That is, you're just measuring the projector on system A and doing the trivial identity measurement on system B. Let's see what this evaluates to:

$$P_i = \langle\psi_i|(|T_{Ai}\rangle\langle T_{Ai}| \otimes I_B\rangle)|\psi_i\rangle = \langle\phi_i^B|\phi_i^B\rangle = \sum_j c_{ij}^*\langle T_{Bj}|\sum_k c_{ik}|T_{Bk}\rangle = \sum_{jk}\delta_{jk}c_{ij}^*c_{ik} = \sum_j |c_{ij}|^2,$$

We can also explicitly compute the normalization of the state after the measurement, which reads

$$|\psi_i'\rangle = \frac{1}{\sqrt{\langle T_{Ai}|T_{Ai}\rangle\langle\phi_i^B|\phi_i^B\rangle}}|T_{Ai}\rangle \otimes |\phi_i^B\rangle = \frac{1}{\sqrt{\sum_j |c_{ij}|^2}}|T_{Ai}\rangle \otimes |\phi_i^B\rangle = \sum_j \frac{c_{ij}}{\sqrt{P_i}}|T_{Ai}\rangle \otimes |T_{Bj}\rangle.$$

2.3.3 Example: Qubit Measurements

Let us consider again an example for two qubits. We consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right),$$

and let us suppose that we are interested in measuring Z on system A. As always, this measurement can yield only two possible outcomes, 1 and -1.

The probability of obtaining +1 on the first qubit is the sum of i. the probability that the first qubit is in the +1 state ($|0\rangle$) *and* the second qubit is in the +1 state ($|0\rangle$) and ii. the probability that the first qubit is in the +1 state ($|0\rangle$) *and* the second qubit is in the -1 state ($|1\rangle$). That is, $P_0 = |c_{00}|^2 + |c_{01}|^2 = \frac{1}{2}$. On getting this outcome, the system then collapses into the normalized state $|0\rangle_A \otimes |1\rangle_B$.

In the other case, i.e., if we get the -1 outcome, it is easy to see that the system collapses into $|1\rangle_A \otimes |0\rangle_B$ also with probability $P_- = \frac{1}{2}$.

2.4 Entanglement

In the previous discussion, we have seen that the measurement of one part of the system directly influences the outcomes of a measurement of the other part. This is one manifestation of what is called quantum "entanglement". More specifically, a state of two systems is said to be entangled if its coefficients cannot be written as the product of two independent coefficients.

If instead, the global wave function can be written as the product of two wave functions corresponding to the subsystems A and B, then we say that the system is "separable". For a separable state, the wave function then reads

$$|\psi\rangle_{\text{sep}} = \sum_{ij} c_{ij}|T_{Ai}\rangle \otimes |T_{Bj}\rangle = \sum_{ij} c_i^{(A)} c_j^{(B)} |T_{Ai}\rangle \otimes |T_{Bj}\rangle = \left(\sum_i c_i^{(A)}|T_{Ai}\rangle\right) \otimes \left(\sum_j c_j^{(B)}|T_{Bj}\rangle\right) = |\psi\rangle \otimes |\phi\rangle.$$

If a system is separable, we also immediately see that a measurement performed on one part does not affect the other one. For example, if we measure T_A, the system will collapse into some state

$$|\psi_i\rangle = |T_{Ai}\rangle \otimes |\phi\rangle,$$

with probability $|c_i^{(A)}|^2$, but the resulting state for the subsystem B will always be $|\phi\rangle$, independently of the outcome of the measurement on A.

To explicitly determine whether a state is separable or entangled, we have to check whether the matrix of coefficients factorizes or not, namely if the condition $c_{ij} = c_i^{(A)} c_j^{(B)}$ is verified or not. For example, for two qubits, the condition of separability reads

$$\begin{aligned} c_{00} &= c_0^{(A)} c_0^{(B)}, \\ c_{01} &= c_0^{(A)} c_1^{(B)}, \\ c_{10} &= c_1^{(A)} c_0^{(B)}, \\ c_{11} &= c_1^{(A)} c_1^{(B)}, \end{aligned}$$

These conditions are satisfied if

$$c_{00}c_{11} - c_{01}c_{10} = c_0^{(A)} c_0^{(B)} c_1^{(A)} c_1^{(B)} - c_0^{(A)} c_1^{(B)} c_1^{(A)} c_0^{(B)} = 0,$$

Equivalently, we can write $\det \hat{c} = 0$ where we have conveniently arranged the coefficients c_{ij} in a matrix:

$$\hat{c} = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}.$$

Thus, if the determinant of the coefficient matrix for the state in the composite basis is zero, then the state is separable.

For example, our state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B \right),$$

has a coefficient matrix

$$\hat{c} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

whose determinant is non-zero, and it is thus an entangled state.

Right, that's the basic mathematical formalism you need to be familiar with in order to study the behaviour of composite systems. Let's move onto something more exciting and look at the consequences of this formalism.

2.5 The Quantum Eraser

The two slit experiment is often the first thought experiment a student encounters when studying quantum mechanics. Here we will explore some variants to it that highlight the curious interplay between coherence, interference and entanglement.

Standard two slit experiment (1): Let us start with the standard two slit experiment. We suppose that single horizontally polarized photons impinge on a screen with two slits and hit a second screen placed behind the first (see Fig. 2.1a)). Although the photons hit the screen one by one we see an interference pattern on the screen behind.

Standard two slit experiment (2): We now suppose that a 90 degrees polarisation shifter is placed behind one of the slits (so that the light coming through it now is vertically polarized) but otherwise leave the set up unchanged (Fig. 2.1b). *What happens this time?*

In this case the interference pattern does not arise. Instead we see a simple mixture of the two patterns we would get if the photons went either through the top or the bottom slit as shown in Fig. 2.1b. This is because if we measured each photons polarisation then we would be able to determine if it went through the top or the bottom slit. Even if we do not in fact check which slit we went through this information is enough to destroy the interference pattern.

Here is how to understand this mathematically. Let $\psi_1(x,t)$ be the wavefunction of a photon emerging from the first slit, and $\psi_2(x,t)$ be that from the second slit. Let the polarisation of a photon be labelled by a H (horizontal) or V (vertical) substate, so that a horizontally-polarised